Antenna Beam Solid Angle Relationships

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Antenna beam solid angle is a critical system parameter whose value needs to be known accurately for some types of radio science experiments and radiometer system evaluations. Methods for determining antenna beam solid angle are not well known because this subject has not been discussed extensively in technical literature. This article fulfills the need for a summary of useful formulas and methods for determining antenna beam solid angle.

I. Introduction

A method used by Franco et al. (Ref. 1) for verifying the performance of a 20.7/31.4-GHz Water Vapor Radiometer (WVR) system was to experimentally record the system noise temperature during the time that the sun drifted through the peak of the main beam of the WVR horn. Comparison of the measured to the known noise temperature of the sun provided a means for determining the accuracy of the WVR system. The main drawback to this particular method is that it is necessary to know the precise values of the antenna beam solid angle of the WVR horn for the frequencies at which the drift curve data were obtained. It was originally believed that knowledge of the antenna half-power beamwidths was sufficient information for computing antenna beam solid angle. This assumption turned out to be incorrect for the general case and this discovery made it necessary to determine antenna beam solid angle by another method discussed in this article.

The purpose of this article is to present some of the useful formulas and methods for determining antenna beam solid angle. To this author's knowledge, these formulas cannot be readily found in well-known textbooks or publications and, therefore, this article should be useful to experimenters performing similar types of radiometer system evaluations in the future.

II. General Case Antenna Relationships

In order to show the relationships of beam solid angle to such antenna parameters as half-power beamwidths and antenna gain, it is helpful to begin with some basic definitions given by Kraus in Ref. 2. Antenna beam solid angle Ω_A is the angle through which all the power from a transmitting antenna would stream if the power (per unit solid angle) were constant and equal to the maximum value. A pictorial description of this definition may be helpful and is depicted in Fig. 1. Expressed mathematically

$$\Omega_A = \int_0^{2\pi} \int_0^{\pi} P_n(\theta,\phi) \sin\theta d\theta d\phi \tag{1}$$

where (θ, ϕ) are spherical coordinate angles and $P_n(\theta, \phi)$ is the antenna power pattern normalized to the maximum value and expressed mathematically as

$$P_n(\theta,\phi) = \frac{P(\theta,\phi)}{P_{max}} \tag{2}$$

The main beam or main lobe solid angle Ω_M is expressed as

$$\Omega_M = \iint_{\substack{main\\beam}} P_n(\theta,\phi) \sin\theta \, d\theta \, d\phi \tag{3}$$

Beam efficiency is determined from

$$\epsilon_M = \frac{\Omega_M}{\Omega_A} = \left[\frac{\displaystyle\iint_{\substack{main\\beam}} P(\theta,\phi) \sin\theta \, d\theta \, d\phi}{\displaystyle\iint_{\substack{all\\space}} P(\theta,\phi) \sin\theta \, d\theta \, d\phi} \right] \tag{4}$$

It follows that

$$\Omega_A = \frac{\Omega_M}{\epsilon_M} \tag{5}$$

It is often convenient to relate main beam solid angle to the antenna half-power beamwidth, i.e., the full beamwidth between the 3-dB points. In Ref. 2, the relationship is given by Kraus as

$$\Omega_M = k_p \theta_{HP} \phi_{HP} \tag{6}$$

where

k_p = factor between about 1.0 for uniform aperture distribution and 1.13 for a Gaussian power pattern

θ_{HP} = full beamwidth between half-power points on the θ-plane power pattern, rad

ϕ_{HP} = full beamwidth between half-power points on the ϕ-plane power pattern, rad

Substitution of Eq. (6) into Eq. (5) results in

$$\Omega_A = \frac{k_P}{\epsilon_M} \theta_{HP} \phi_{HP} \tag{7}$$

which is a relationship not found in Kraus (Ref. 2) or Ko (Ref. 3). For an example application of the above formula, consider a large antenna with uniform aperture illumination. Substitutions of $k_P = 1.0$ and $\epsilon_M = 0.75$ values given by Kraus (Ref. 2) into Eq. (7) give the special case result of

$$\Omega_A = \frac{4}{3} \theta_{HP} \phi_{HP}$$

The above formula given by Eq. (7) is useful when the values of k_P and ϵ_M for the horn or antenna are known or have been previously determined. In the general case, however, these values are not known and not easy to determine except for very special cases. A more direct approach to finding Ω_A is desirable.

A second method for finding Ω_A is to obtain theoretical or experimental far-field patterns of the antenna at the desired frequencies of operation. Then one could use the fundamental definition of antenna beam solid angle given by Eq. (1) and perform a numerical integration of the normalized pattern. The pattern integrations can be carried out through the use of numerical methods similar to those used in JPL antenna computer programs (Ref. 4). The disadvantage of this method might be that antenna pattern data, except in the case of small horns, might be difficult to obtain experimentally.

A third method for determining Ω_A is to use knowledge of maximum directive gain of the antenna. The directive gain is given as

$$D(\theta,\phi) = \frac{P_n(\theta,\phi)}{\frac{1}{4\pi}\displaystyle\int_0^{2\pi}\int_0^{\pi} P_n(\theta,\phi) \sin\theta \, d\theta \, d\phi} = \frac{4\pi}{\Omega_A} P_n(\theta,\phi) \tag{8}$$

where

$$P_n(\theta,\phi) = \frac{P(\theta,\phi)}{P_{max}} = \text{normalized power pattern}$$

Then at $\theta = \theta_0$, $\phi = \phi_0$ corresponding to $P(\theta,\phi) = P_{max}$

$$P_n(\theta_0,\phi_0) = \frac{P(\theta_0,\phi_0)}{P_{max}} = 1$$

and the *maximum* directive gain is

$$D(\theta_0,\phi_0) = D_M = \frac{4\pi}{\Omega_A}$$

so that

$$\Omega_A = \frac{4\pi}{D_M} \tag{9}$$

Note that $D(\theta, \phi)$ is based entirely on *antenna pattern data* and that loss information is not required. The directive antenna gain differs from the actual antenna gain in that the latter includes losses in the antenna structure and feed. Fortunately in the case of most WVR horns, the mismatch and waveguide losses are small and little accuracy would be lost if the actual antenna gain value were substituted for directive gain in Eq. (9). Antenna gain is a value that is generally known and, if not known, experimental techniques for determining maximum antenna gain are well documented.

It is of interest at this point to derive a more general expression for k_p, which is a constant used by Kraus and Ko for classifying antennas with various types of aperture illuminations. From previous equations given above

$$k_p = \frac{\Omega_M}{\theta_{HP}\,\phi_{HP}}$$

$$\Omega_M = \Omega_A\,\epsilon_M = \frac{4\pi\,\epsilon_M}{D_M}$$

so that

$$k_p = \frac{4\pi\,\epsilon_M}{D_M\,\theta_{HP}\,\phi_{HP}} \tag{10}$$

III. Check Case

In order to verify the relationships given in this article, calculations were performed for a circular aperture antenna with uniform aperture illumination. As given in Silver (Ref. 5), the normalized far-field pattern is of the form

$$P_n(\theta) = \frac{P_E(\theta)}{P_{max}} = \frac{P_H(\theta)}{P_{max}} = |\Lambda_1(u)|^2 \tag{11}$$

where $P_E(\theta)$ and $P_H(\theta)$ are E- and H-plane power patterns, and

$$\Lambda_1(u) = \frac{2J_1(u)}{u} \qquad \text{for } 0 \leqslant \theta \leqslant \frac{\pi}{2}$$
$$= 0 \qquad \text{for } \frac{\pi}{2} < \theta \leqslant \pi \tag{12}$$

$J_1(u)$ = Bessel function of the first kind, first order

$$u = \frac{\pi d}{\lambda} \sin\theta$$

and d is the antenna aperture diameter and λ_0 is the operating free space wavelength. To perform a check case, a value of $d/\lambda_0 = 8.5732$ was selected because it results in a half-power beamwidth value identical to that for one of the WVR horns used in a radiometer performance evaluation (Ref. 1). Theoretical pattern data were calculated from the above equation and then used as input data for a JPL Efficiency Program of the type described in Ref. 4. The resulting values from the computer program were

$$\epsilon_M = 0.84028$$

$$\theta_{HP} = 0.1201 \text{ rad} \quad (6.881 \text{ deg})$$

$$\phi_{HP} = 0.1201 \text{ rad} \quad (6.881 \text{ deg})$$

$$10 \log_{10} D_M = 28.61 \text{ dB}$$

which agree with values of gain and half-power beamwidths directly calculated from exact formulas given in Silver (Ref. 5) for this special case. Substitution of these computed values into the above Eq. (10) resulted in a value of $k_p = 1.008$, which is in exact agreement with the value published by Ko (Ref. 3) for the uniform illumination case.

Agreement with Ko's value of $k_p = 1.13$ was also obtained for the case of calculated far-field patterns for a Gaussian aperture illumination.

IV. Conclusions

General case relationships for determining antenna beam solid angle have been derived and presented. If the maximum antenna gain of the antenna is known, then the expression given by Eq. (9) is the easiest one to use for computing antenna beam solid angle. The relationships given in this article should prove useful to experimenters desiring to use a drift curve method to evaluate radiometric system performance.

References

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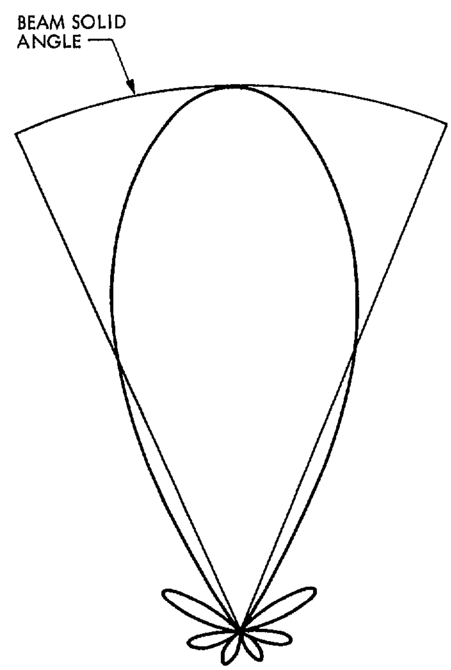

Fig. 1. Relation of antenna beam solid angle to antenna pattern